Sec. 11.5 The Short Run Behavior of Rational Functions

Analyzing the Graph of a Rational Function –

1. Find the domain of the rational function.
2. Write R in lowest terms.
3. Locate the intercepts of the graph. Remember that the x-intercepts of the numerator are the x-intercepts of the entire function. The y-intercept (if there is one) will be $R(0)$.
4. Locate the vertical asymptotes.
5. Locate the horizontal or oblique asymptotes.
6. Determine the points, if any, at which the graph of R intersects the asymptotes.

** The graph of a rational function never crosses a vertical asymptote. However, the graphs of some rational functions cross their horizontal asymptotes. The difference is that a vertical asymptote occurs where the function is undefined, so there can be no y-value there, whereas a horizontal asymptote represents the limiting value of the function as x →± ∞.

7. Graph R using a graphing calculator.
8. Use the results obtain in steps 1 through 7 to graph R by hand.

Ex: Describe the end behavior and find the zeros, the y-intercept, and any asymptotes of:

$$R(x) = \frac{x+1}{x(x+4)} = \frac{x+1}{x^2+4x} \quad \lim_{x\to\infty} \frac{x}{x^2} = \frac{1}{x} = 0 \quad \lim_{x\to-\infty} \frac{1}{x} = 0 \Rightarrow HA: \boxed{y=0}$$

Zeros: $x+1=0$

$\boxed{x=-1}$

y-int: $R(0) = \frac{0+1}{0(0+4)} = \frac{1}{0}$

$\boxed{\text{NO Y-INTERCEPT}}$

VA: $x(x+4)=0$

$\boxed{x=0}$ $\quad x+4=0$

$\boxed{x=-4}$

Ex: Describe the end behavior and find the zeros, the y-intercept, and any asymptotes of:

$$R(x) = \frac{x^3-1}{x^2-9} = \frac{(x-1)(x^2+x+1)}{(x+3)(x-3)} \quad \lim_{x\to\infty} \frac{x^3}{x^2} = x = \infty \quad \lim_{x\to-\infty}$$

ZEROS: $x^3-1=0$

$x^3=1$

$\boxed{x=1}$

y-int: $R(0) = \frac{0^3-1}{0^2-9} = \frac{-1}{-9}$

$\boxed{\left(0, \frac{1}{9}\right)}$

VA: $x+3=0$ $\quad x-3=0$

$\boxed{x=-3}$ $\quad \boxed{x=3}$

Ex: Describe the end behavior and find the zeros, the y-intercept, and any asymptotes of:

$$R(x) = \frac{3x^2 - 3x}{x^2 + x - 12}. = \frac{3x(x-1)}{(x+4)(x-3)}$$

$\lim_{x \to \pm\infty} \frac{3x^2}{x^2} = \lim_{x \to \pm\infty} 3 = \boxed{3} \Rightarrow$ HA: $\boxed{y = 3}$

ZEROS: $3x(x-1) = 0$

$3x = 0 \quad x - 1 = 0$

$\boxed{x = 0 \quad x = 1}$

VA: $\boxed{x = -4 \quad x = 3}$

y-int: $R(0) = \frac{3 \cdot 0(0-1)}{(0+4)(0-3)}$

$\frac{0}{-12} = 0$

$\boxed{(0,0)}$

Hole – when simplifying graph, a factor cancels out in the numerator and denominator, creating a new function that resembles the simplified function, but doesn't exist at a single point where a vertical asymptote would have been in the original function.

Ex: $R(x) = \frac{x^2 + x - 12}{x^2 - x - 6}.$ $\frac{(x+4)\cancel{(x-3)}}{(x+2)\cancel{(x-3)}} = \frac{x+4}{x+2}$

HA: $\lim_{x \to \pm\infty} \frac{x}{x} = 1$ $\boxed{y = 1}$

VA: $\boxed{x = -2}$

HOLE: $x = 3$

$\frac{3+4}{3+2} = \frac{7}{5}$

HOLE $\boxed{(3, 1\tfrac{2}{5})}$

y-int: $\frac{0+4}{0+2} = 2$

$\boxed{(0,2)}$

{x-int} ZERO: $x + 4 = 0$

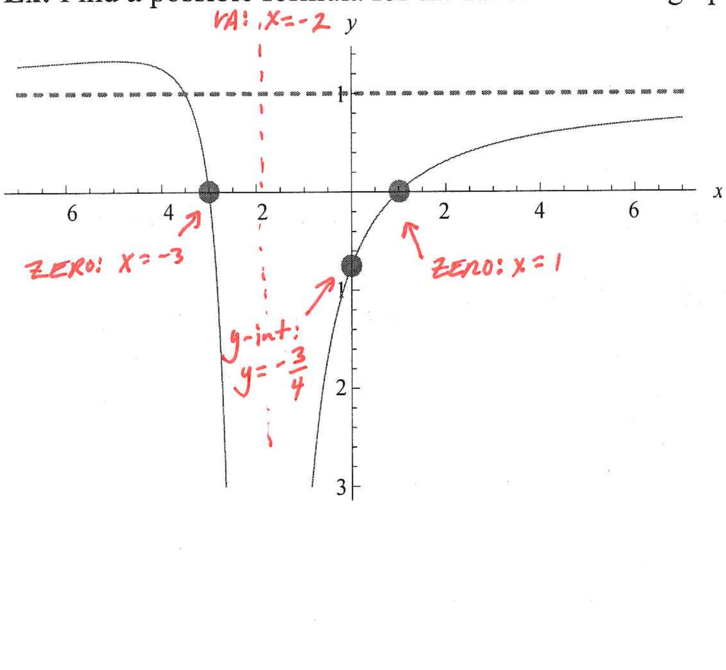
$\boxed{x = -4}$

NOT AT $x = 3$!

__Finding a Formula for a Rational Function from its Graph:__

The graph of a rational function can give a good idea of its formula. Zeros of the function correspond to factors in the numerator and vertical asymptotes correspond to factors in the denominator.

Ex: Find a possible formula for the function whose graph is given below.



HA: $y = 1$

ZEROS: $x = 1$, $x = -3 \Rightarrow \frac{(x-1)(x+3)}{(x+2)}$

VA: $x = -2$

BUT HA at $y = 1$ means degree of numerator and denominator must be the same, so:

$$\boxed{y = \frac{(x-1)(x+3)}{(x+2)^2}}$$

Ex: Find a possible formula for a function that has vertical asymptotes at $x = -2$ and $x = 3$. It has a horizontal asymptote at $y = 1$. The graph of g touches the x-axis once at $x = 5$.

ZERO: $x = 5$ (Double) $\Rightarrow \dfrac{(x-5)^2}{(x+2)(x-3)}$

VA: $x = -2$ $x = 3$

$\lim_{x \to \infty} \dfrac{x^2}{x^2} = 1$ (Horizontal Asymptote)

$$\boxed{g(x) = \frac{(x-5)^2}{(x+2)(x-3)}}$$

Ex: Find a possible formula for a function that has the same characteristics as the previous example except that the horizontal asymptote of h is at $y = 0$.

$$\boxed{h(x) = \frac{(x-5)^2}{(x+2)^2(x-3)} \quad \text{or} \quad h(x) = \frac{(x-5)^2}{(x+2)(x-3)^2}}$$

HW: pg 465-468 #4-6,8,9,15,22,26,36,38,39,40,42